Preprint UTM 705 - Oct 2006

On localization and position operators in Möbius covariant theories.

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Abstract. Some years ago it was shown that, in some cases, a notion of locality can arise from the group of symmetry enjoyed by the theory [1, 2, 3], thus in an intrinsic way. In particular, when Möbius covariance is present, it is possible to associate some particular transformations to the Tomita Takesaki modular operator and conjugation of a specific interval of an abstract circle. In this context we propose a way to define an operator representing the coordinate conjugated with the modular transformations. Remarkably this coordinate turns out to be compatible with the abstract notion of locality. Finally a concrete example concerning a quantum particle on a line is also given.

1 Introduction

In relativistic quantum theories, localization is usually introduced only at the level of second quantization, considering the smearing of field operators by means of real local functions. In particular, the set of locally smeared free fields generates a suitable $*$-algebra of operators, moreover operator smeared by functions with spatially separated domain have to commute. We stress that local operators are characterized by the corresponding real local smearing wavefunctions, that generate a dense real subspaces, $\mathbb{R}$-*subspace* of the one particle Hilbert space (of the corresponding free theory). While the characterization of the one particle Hilbert space (Wigner space [4]) uses as building blocks only the abstract symmetry enjoyed by the theory, the definition of local operators, as usually given, requires functions in position representation. Thus the localization procedure, making use of functions in a particular representation, is not completely intrinsic.

For this reason, it was proposed some years ago by Brunetti, Guido and Longo [1] and by Schroer [3, 2] to introduce $\mathbb{R}$-subspaces of local smearing wavefunctions in a more intrinsic way[2]. They suggested to interpret some suitable one-parameter subgroups of the group of symmetry as the modular group introduced by Tomita Takesaki [9]. This modular structure is used to find local subspaces, hence called modular localization. The described procedure, making use only of symmetry properties, is completely intrinsic and no particular Hilbert space

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[2] See also [5, 6, 7, 8].

representation is required. We remind that there is another localization scheme called Newton Wigner (NW) localization [10, 11, 12], whose local objects are found by smearing with elements of proper subspaces of the one particle Hilbert space. But since the NW localization is not fully compatible with relativistic theory, we shall not assume this point of view.

Here we would like to consider quantum theories showing Möbius covariance [13, 14, 15, 16, 17, 18], as usual this means that, on the Hilbert space of the theory $\mathcal{H}$, acts an (anti)-unitary representation $\pi_{\mathcal{M}}$ of the Möbius group $\mathcal{M}$. As Poincaré invariant theories induce modular locality, also in $\mathcal{M}$ invariant theories there is a suitable identification of the (proper) intervals of an abstract circle $\mathbb{S}^1$ with particular subgroups of the representation. In fact, for every interval I, is possible to select the following operators that represent particular Möbius transformations: J_I which is anti-unitary and a one parameter subgroup of unitary transformations Δ_I^{it}. In particular, J_I and Δ_I play the role of Tomita Takesaki modular conjugation and modular operator for the interval I, then reverting the usual point of view, local $\mathbb{R}$-subspaces are defined as

$$\mathcal{K}_I := \{\psi \in \mathcal{H} : J_I \Delta_I^{1/2} \psi := \psi\} .$$

We would like to stress that, in general, the elements of $\mathcal{K}_I$ are not functions with domain contained in the interval I, moreover there is no direct connection between intervals and operators and the geometric interpretation of the action of J_I and Δ_I on elements of the Hilbert space is not requested.

The new aspect we are going to study is the presence of selfadjoint operators, *position operators*, representing suitable coordinates within the abstract circle $\mathbb{S}^1$. Notice that Pauli theorem [19] prevents the existence of a selfadjoint operator for a global coordinate, that should be canonically conjugated with the generator of rotations which is positive. The same obstruction appears for the observable representing time of arrival in quantum theories [20] or for the relativistic coordinates of an event [21]. Such kind of problems are usually addressed generalizing the concept of observable that turns out to be described by means of positive operator valued measures (POVM) [22, 23, 24]. Here we do not want to use this difficult method, we try instead to define only *local coordinate*, choosing an interval I and then defining the coordinate operator for the quantum theory restricted to I. Furthermore it is known that the vacuum theory on the circle appears as a thermal theory when restricted on an interval with respect to modular dilations. We know that in thermal theories selfadjoint operator representing time can be defined [25], than we expect that a selfadjoint operator for coordinate conjugated with modular dilations Δ_I^{it} also exists.

It is known that, in the Newton Wigner localization picture, there are meaningful coordinate operators. The situation appears completely different adopting the modular localization. Then the study of the interplay of the coordinate operators with modular locality seems to be relevant. Since local object are described by dense $\mathbb{R}$-subspace of the one particle Hilbert space and not by proper complex subspaces we do not expect to find the compatibility of coordinate operators T with locality from its spectral properties. Furthermore we shall show an interplay between modular locality and the range of the expectation value of T that turns out to be described by a and b, when T is evaluated in $\mathcal{K}_{[a,b]}$ contained in $\mathcal{K}_I$. Hence the intrinsic locality has not only an abstract meaning, but it turns out to describe the range of the expectation values of the coordinate operator T. In this way it acquires a suitable physical meaning, even if it does not describe a Newton Wigner coordinate. Remarkably, T turns out to be defined by a particular combination of the selfadjoint operators representing the generators of the $PSL(2,\mathbb{R})$ group contained in the Möbius group of symmetry. In particular for the coordinate within the half of the circle I_1, being H, D, C a particular set of generators of $PSL(2,\mathbb{R})$ given below, T is defined as $T = \log(H^{-1/2}CH^{-1/2})/2$ while D generates modular dilations $\Delta_{I_1}^{it}$.

The paper is organized as follows: first of all, some known results are presented in a useful way for the development of the subject. Then in the third section the most important results are presented. The fourth section contains the discussion of a concrete example where locality arises by construction. It is shown that the proposed coordinate turns out to be compatible with the intrinsic modular locality. Then some final comments are presented. In the appendix two kind of irreducible representations of the covering group of $SL(2,\mathbb{R})$ on $L^2(\mathbb{R}^+, dE)$ are discussed.

2 Preliminary considerations

2.1. Interplay between Möbius group, $PSL(2,\mathbb{R})$, modular operator and modular conjugation. The Möbius group is made by conformal transformations of $\mathbb{C}$ leaving the circle $\mathbb{S}^1$ (modulus one elements of $\mathbb{C}$) invariant, it is generated by the group $PSL(2,\mathbb{R})$ and by an involution. We remind that $PSL(2,\mathbb{R})$ is the left coset $SL(2,\mathbb{R})/\{+\mathbb{I}, -\mathbb{I}\}$, where $SL(2,\mathbb{R})$ is made by real two dimensional square matrix g with determinant one satisfying $g\beta g^t = \beta$. β is the anti-diagonal matrix whose anti-diagonal elements are $(1,-1)$ (from top to bottom). As already said, the circle $\mathbb{S}^1$ can be seen as the modulus one elements of $\mathbb{C}$ but also as the compactified line $\mathbb{R}\cup\{\infty\}$, and the two picture are connected by the Caylay transform $C : z \mapsto -i(z+1)(z-1)^{-1}$ that maps the elements of the circle to the compactified line. The action of $PSL(2,\mathbb{R})$ on $\mathbb{S}^1$ descends from the action of $PSL(2,\mathbb{R})$ on $\mathbb{R}\cup\{\infty\}$:

$$x \to \frac{ax+b}{cx+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R}). \tag{1}$$

$g \in PSL(2,\mathbb{R})$ can be decomposed in the following way (Iwasawa decomposition) :

$$g := T(x)\Lambda(y)P(z), \qquad x, y, z \in \mathbb{R}, \tag{2}$$

where

$$T(x) := e^{xh} := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad \Lambda(y) := e^{yd} := \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \qquad P(z) := e^{zc} := \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}.$$

h, c, d form a basis of the Lie algebra $sl(2,\mathbb{R})$ and they satisfy following commutation relations:

$$[h, d] = h, \qquad [c, d] = -c, \qquad [c, h] = 2d\ . \tag{3}$$

The other element generating Möbius covariance is the reflection r that maps x to $-x$ in $\mathbb{R}\cup\{\infty\}$.

In the subsequent part of the paper we shall use the representation of $\mathbb{S}^1$ as the compactified line. Furthermore, we shall be interested in the set $\mathcal{I}$ of proper intervals of $\mathbb{S}^1$. Remind that an interval I is *proper* if the interior of I and is complement I' are not empty. Every element I of $\mathcal{I}$ is denoted by $[a, b]$ where a and b are the starting and ending point in $\mathbb{R} \cup \{\infty\}$. The half circle will be denoted by I_1, the particular set $[0, +\infty]$. The action of $\Lambda(x)$ is closed in I_1, while the r map I_1 to its complement I_1' in $\mathbb{S}^1$.

Proposition 2.1. *For every proper interval I exists a $g \in PSL(2, \mathbb{R})$ such that $I = gI_1$, moreover the Iwasawa decomposition (2), can be generalized to every interval I being $T_I(a) = gT(a)g^{-1}$, $P_I(a) = gP(a)g^{-1}$ and $\Lambda_I(a) = g\Lambda(a)g^{-1}$. Furthermore $r_I = grg^{-1}$ maps I in I'*

The following properties, that can be checked by (1) and by the preceding proposition, hold for the general interval I:

(A) Reflection covariance: r_I maps I to I' and $r_{gI} = gr_I g^{-1}$ for every $g \in PSL(2, \mathbb{R})$.

(B) Λ covariance: The action of Λ_I is closed in I and $\Lambda_{gI}(t) = g\ \Lambda_I(t)\ g^{-1}$. for g in the triangular subgroup.

(C) Positive inclusions: We have two type of positive inclusions, in fact the action of $T_I(t)$ is closed in I for positive t and that satisfy the following relation:

$$\Lambda_I(b)T_I(t)\Lambda_I(-b) := T_I(e^{2\pi b}t);$$

while the action of $P_I(t)$ is closed in I for negative t and that satisfy the following relation:

$$\Lambda_I(b)P_I(t)\Lambda(-b) := P_I(e^{-2\pi b}t).$$

As a final remark we would like to remember that there is another particular one parameter subgroup of transformation $R(\theta)$ generated by $(h+c)/2$ whose action on point is that of rotating the circle $\mathbb{S}^1$ by an angle θ. In particular $R(\pi)$ maps the interval I_1 to its complement I_1', x in $-1/x$ and

$$R(\pi)hR(-\pi) = c \qquad R(\pi)dR(-\pi) = -d \qquad R(\pi)cR(-\pi) = h\ .$$

2.2. Quantization and locality: Weyl algebras, von Neumann algebras and conformal net. Up to now we have described only an abstract connection between a representation of the Möbius group and intervals of a circle. Now we would like to analyze its interplay with a quantum theory. Hence we consider a proper positive energy (anti)-unitary representation U of the Möbius group on the one particle separable Hilbert space $\mathcal{H}$ of the considered quantum system. In particular U contains an unitary positive energy representation of $PSL(2, \mathbb{R}) \subset \mathcal{M}$. Then passing to the representation of the corresponding $sl(2, \mathbb{R})$ algebra, h, d, c (3) turns out to be represented by the self-adjoint operators H, C and D, enjoying the following commutation relations:

$$[H, D] = iH, \qquad [C, D] = -iC, \qquad [H, C] = 2iD. \tag{4}$$

We will denote respectively by $\mathcal{D}(H), \mathcal{D}(D), \mathcal{D}(C)$ the domain of selfadjointness of H, D and C. Furthermore, since U contains a unitary representation of $PSL(2,\mathbb{R})$ there exists a common dense set of analytic vectors for H, D and C. Thus the analytic vectors are contained in $\mathcal{D} := \mathcal{D}(H) \cap \mathcal{D}(D) \cap \mathcal{D}(C)$ which turns out to be a dense set in $\mathcal{H}$. Explicit representations of H, D, C in the case of irreducible representations can be found in appendix A. Notice that H and C are positive operator while the spectrum of D covers the whole $\mathbb{R}$. To implement the properties (a) (b) and (c) described above we need to build a local theory, in other word we would like to find real subspaces of $\mathcal{H}$ representing local objects. The first step is to define the Tomita operators associated with intervals I, in term of the unitary representation of dilations $\Lambda_I(t) \subset PSL(2,\mathbb{R})$ by means of Δ_I^{it}, and the anti-unitary representation of reflections r_I by J_I. Then we shall define S_I as

$$S_I := J_I \Delta_I^{1/2} .$$

By means of S_I we can define the real subspace of "local functions"

$$\mathcal{K}_I := \{\psi \in \mathcal{H} : S_I \psi = \psi\} ; \tag{5}$$

(considering I_1, J_{I_1} maps $\mathcal{K}_{I_1}$ in its complement and $\Delta_{I_1}^{it/(2\pi)} := e^{-itD}$). On $\mathcal{K}_I$ the imaginary part of the scalar product of $\mathcal{H}$ defines a symplectic form σ.

Proposition 2.2. *σ and $\mathcal{K}_I$, quotiented with the null space of σ, defines a symplectic structure over $\mathcal{H}$.*

Notice that $\mathcal{H}$, the target space of the unitary representation, needs to be interpreted as the one-particle Hilbert space.

Weyl algebra. The concrete Weyl quantum fields $\hat{W}(\psi) := e^{i\hat{\varphi}(\psi)}$ on the standard Fock space $\mathfrak{F}(\mathcal{H})$ generates a unitary representation π of the Weyl algebra $\mathfrak{A}(I)$ associated with the pair $(\mathcal{K}_I, \sigma_I)$, called vacuum representation, in fact it is nothing but the GNS representation related with the vacuum state Ω. $\hat{\varphi}(\psi)$ is the symplectically smeared field defined as $\hat{\varphi}(\psi) := ia(\psi) - ia^\dagger(\psi)$, $\psi \in \mathcal{K}_I$. $a(\psi)$ and $a^\dagger(\psi)$ being creation and annihilation operator of the state ψ.

Proposition 2.3. *(locality) Because of the symplectic form σ a locality principle holds: If I_a and I_b are disjoint subintervals of I, in $\mathfrak{A}(I)$, we have*

$$[\hat{W}(\psi), \hat{W}(\psi')] = 0, \qquad \textit{if} \qquad \psi \in \mathcal{K}_{I_a},\ \psi' \in \mathcal{K}_{I_b}.$$

von Neumann algebra. Then the von Neumann algebra $\mathfrak{M}(I)$ associated with the interval I over $\mathfrak{H} = \overline{\mathfrak{F}(\mathcal{H})}$ can be defined as

$$\mathfrak{M}(I) := \{W(\psi) : \psi \in \mathcal{K}_I\}'',$$

Notice that the group representations translates in a simple way to the von Neumann algebra through the action on $\psi \in \mathcal{K}_I$.

Conformal Net The last structure we would like to discuss here is the conformal net that arises considering the set of all local conformal algebras $\mathfrak{M}(I)$ for every interval I. Notice that the following properties are satisfied by this set:

(A) Isotony. If $I_1 \subset I_2$ then $\mathfrak{M}(I_1) \subset \mathfrak{M}(I_2)$.

(B) Locality. If I_1 and I_2 are disjoint proper intervals, then $\mathfrak{M}(I_1) \subset \mathfrak{M}(I_2)'$.

(C) Conformal invariance. There exists a strongly continuous unitary representation U of $PSL(2,\mathbb{R})$ on $\mathfrak{H}$ such that $U(g)\mathfrak{M}(I)U(g)^* = \mathfrak{M}(gI)$,$g \in PSL(2,\mathbb{R})$.

(D) Positivity of the energy. Being $R(\theta)$ the rotation of an angle θ on $\mathbb{S}^1$, the generator of the rotation subgroup $U(R(\theta))$ (also called conformal Hamiltonian) is positive.

(E) Existence of the vacuum. There exists a unit vector $\Omega \in \mathfrak{H}$ (vacuum vector) which is invariant under the unitary representation of $PSL(2,\mathbb{R})$ and cyclic for every $\mathfrak{M}(I)$ where I is a proper interval.

A consequence of the presence of a conformal net is the Reeh-Schlieder property [14, 15, 16, 18]. In particular, it says that the the vacuum vector Ω is cyclic and separating for any local algebra $\mathfrak{M}(I)$, hence, for every $\mathfrak{M}(I)$, Ω is a KMS state with respect to the modular group Δ_I^{it}. In particular for the case of the von Neumann algebra build over half of the circle $\mathfrak{M}_{I_1}$, Ω can be interpreted as a thermal state at temperature $1/(2\pi)$ with respect to the unitary transformation generated by D. D can be interpreted as the energy of the particular system.

As a final comment we notice that $\mathcal{K}_I \subset \mathcal{H} \subset \mathfrak{H}$ and that the Hilbert subspace $\mathcal{H}$ does not contain Ω, furthermore $\mathcal{K}_I := P\mathfrak{M}(I)\Omega$, where P projects elements of $\mathfrak{H}$ in $\mathcal{H}$. In the next part of the paper we shall use only the fact that being $\psi = A\Omega$ with $A \in \mathfrak{M}(I)$ then $J\Delta_I^{1/2}\psi = \psi$. And the fact that H and C have continuous spectrum on $\mathcal{H}$, hence the result resented below for $\mathcal{K}_I \subset \mathcal{H}$ can be generalized to $(\mathfrak{M}(I)\Omega \cap \{\Omega\}^{\perp}) \subset \mathfrak{H}$.

3 Modular coordinate

The aim of the present section is to find operators interpretable as coordinates of intervals in the case of theories invariant under Möbius group. We notice immediately that a global coordinate of $\mathbb{S}^1$ cannot be represented by a self-adjoint operator, in fact such an operator should be conjugate to the generator of rotations which exists in the algebra of observables and it is bounded from below. In this case Pauli theorem [19] prevent the existence of a conjugated selfadjoint operator. This problem can be circumvented generalizing the concept of observables by means of POVM [22, 23, 24]. Here we would like not to address the problem in this way. We are searching for coordinates of intervals whose corresponding translation are described by the modular group Δ_I^{it} and whose corresponding generator, is not bounded below. From now on we fix, without losing the generality, a specific interval I_1 which correspond to half of the circle or equivalently to half of the line in the projective representation. The associated modular operator is $\Delta := e^{-2\pi D}$.

Where D is the generator of dilation described above. Then we would like to find a generator T of a one group of unitary transformation $W(a)$ that forms a Weyl Heisenberg group with $\Delta^{it/(2\pi)}$:

$$\Delta^{it/(2\pi)}W(a) = e^{iat}W(a)\Delta^{it/(2\pi)}. \tag{6}$$

Notice that the operator we are looking for should have a continuous spectrum that coincides with $\mathbb{R}$. Then an important step in our work is to check if the found operator when evaluated on wave-function local in $I = [a, b] \subset I_1$ gives results contained in the real interval $(\log a, \log b)$. If such an operator exists it could be interpreted as the modular coordinate for the interval I_1.

Let's start with the following proposition, that will also be useful later:

Proposition 3.1. *Suppose that on $\mathcal{H}$ acts a positive energy unitary representation of the covering group of $SL(2,\mathbb{R})$, generated by H, C, D. Which are selfadjoint operator which satisfy (4), on $\mathcal{D} := \mathcal{D}(H) \cap \mathcal{D}(C) \cap \mathcal{D}(D)$. Consider the self-adjoint operator $H^{-1/2}$ with domain $\mathcal{D}(H^{-1/2})$ defined via spectral theorem. Then $\mathcal{D} \subset \mathcal{D}(H^{-1/2})$. Let $\psi \in \mathcal{D}$ then*

$$\|H^{-1/2}\psi\|^2 \leq \gamma(\psi, C\psi)$$

where γ is a positive number.

Proof. Decompose the Hilbert space $\mathcal{H} = \bigoplus_i \mathcal{H}_i$, where the subspaces $\mathcal{H}_i$ are $PSL(2,\mathbb{R})$ irreducible. The irreducible representation on $\mathcal{H}_i$ is individuated by k_i: the lowest eigenvalue of $(H + C)/2$ on $\mathcal{H}_i$. Then k is the lowest k_i. Since we are considering positive energy representation of $PSL(2,\mathbb{R})$ every $k_i \geq 1$, hence also $k \geq 1$. Being $\gamma = 1/(k - 1/2)^2$ the following holds

$$\|DH^{-1/2}\psi\|^2 + \left(k - \frac{1}{2}\right)^2 \|H^{-1/2}\psi\|^2 \leq \gamma(\psi, C\psi),$$

hence the thesis □

Remark: In a similar way it can be proved that $\mathcal{D}$ is also contained in the domain $\mathcal{D}(C^{-1/2})$ of selfadjointness of $C^{-1/2}$.

The positive inclusions, generated by h and c, introduced at group level are also present as unitary transformation of the Hilbert space.

Proposition 3.2. *Given a positive energy unitary representation of $PSL(2,\mathbb{R})$ on $\mathcal{H}$, be $U_h(a)$ and $U_c(b)$ the two one parameter unitary subgroup generated by H and C then:*
(a) $U_h(a)\mathcal{K}_{I_1} \subset \mathcal{K}_{I_1}$ for every strictly positive a and

$$\Delta^{it}U_h(a)\Delta^{-it} = U_h(e^{-2\pi t}a)\forall t \in \mathbb{R} \qquad JU_h(a)J = U_h(a)^*;$$

(b) $U_c(a)\mathcal{K}_{I_1} \subset \mathcal{K}_{I_1}$ for every strictly negative a and

$$\Delta^{it}U_c(a)\Delta^{-it} = U_c(e^{2\pi t}a)\forall t \in \mathbb{R} \qquad JU_c(a)J = U_c(a)^*;$$

From the positive inclusion properties enjoyed by the representation of $PSL(2,\mathbb{R})$ is possible to build a one-parameter group of unitary transformation forming a Weyl Heisenberg group with the modular transformation Δ^{it}, in fact the following theorem holds.

Theorem 3.1. *Under the hypothesis of proposition 3.2, two self-adjoint operators T_h and T_c exist. Moreover they generates two one parameter groups of unitary transformations forming, with the modular dilations, two Weyl-Heisenberg groups.*

Proof. $V(t) := \Delta^{it/(2\pi)}$ acts as a one-parameter group of unitary transformations on $\mathcal{H}$, To find the other group $W(a)$ we study the properties of H, the generators of $U_h(a)$. From the proposition 3.2 $\Delta^{it} H \Delta^{-it} = e^{-2\pi t} H$, it implies for the positive spectrum $\sigma(H)$ of H that $\sigma(H) = e^{-2\pi t}\sigma(H)$. On $\mathcal{H}$, since the representation we are considering is non trivial, the spectrum has only an orbit $(0, +\infty)$. The spectral decomposition of H on $\mathcal{H}$ is $H := \int_{\mathbb{R}+} \lambda dE(\lambda)$, then the operator

$$T_h := \log(H) = \int_{\mathbb{R}+} \log(\lambda) dE(\lambda) \ , \qquad \text{on} \qquad \mathcal{D}(T_h) := \left\{ \psi : \int_{\mathbb{R}+} |\log(\lambda)|^2 (\psi, dE(\lambda)\psi) \right\} \ ,$$

is selfadjoint and $\mathcal{D}(T_h)$ is dense in $\mathcal{H}$. Let $W(a) := e^{iaT_h}$, from proposition 3.2 descends that $V(t)W(a) := e^{ita}W(a)V(t)$, forming a projective unitary representation of Weyl-Heisenberg group. The same holds also considering $T_c := \log C$. $\square$

Then because of proposition 3.1 and because of spectral properties descend the following

Corollary 3.1. *Under the hypotheses of theorem 3.1, $\mathcal{D} \subset \mathcal{D}(T_h)$ and $\mathcal{D} \subset \mathcal{D}(T_c)$.*

Within the group representation there are two possible operators enjoying the canonical commutation relation with D. By means of them it is possible to find plenty of candidate for describing a coordinate of I_1. In fact every $T_x := x\log(C) - (1-x)\log(H) + f(D)$ with $0 \leq x \leq 1$, where $f(D)$ is an almost general function, if it is self-adjoint on a dense domain $\mathcal{D}(T_x)$, with D generates a Weyl Heisenberg group. Thus the commutation relations alone cannot give a particular meaningful coordinate operator. In the next sub-section we shall study a criterion to choose a preferred one.

3.1. Interplay with locality. We have seen that for every interval there are several possible different position operators. A good criterion to chose a preferred one is to study the compatibility with locality. Since, for systems presenting Möbius covariance, locality arises intrinsically from the group properties and since also the coordinate operators descend from the group generators itself we would like that the choice we are going to do be completely driven by the (anti)-unitary representation of Möbius group.

In particular we say that a selfadjoint operator T_x is compatible with locality if considering $I = [a, b] \subset I_1$, its expectation values on $\mathcal{K}_I$ are between $\log(a)$ and $\log(b)$. We start with some preliminary propositions. From now one $\mathcal{K} := \mathcal{K}_{I_1}$.

Proposition 3.3. *Let D be the generator of dilation of an unitary representation of $SL(2,\mathbb{R})$, and $\mathcal{K}$ as described above. Then D is positive on $\mathcal{K} \cap \mathcal{D}(D)$.*

Proof. Since $\psi \in \mathcal{K}$, $J\Delta^{1/2}\psi = \psi$, then $\|D\psi\| = \|D\Delta^{1/2}\psi\|$, but $\psi \in \mathcal{D}(D)$ then also $\Delta^{1/2}\psi \in \mathcal{D}(D)$. Be α in the strip $\mathcal{S} := \{z \in \mathbb{C} : 0 < \Re(z) < 1\}$, by spectral calculus

$$\|D\Delta^{\alpha/2}\psi\|^2 \leq \|D\psi\|^2 + \|D\Delta^{1/2}\psi\|^2,$$

which means that $\Delta^{\alpha/2}\psi \in \mathcal{D}(D)$ with $\alpha < 1$. Consider the function $F(\alpha) := (\psi, D\Delta^{\alpha}\psi)$, it is analytic in the strip $\mathcal{S}$ and in particular it is real and smooth for α real and $0 < \alpha < 1$. Notice that $|F(0)| = |(\psi, D\psi)|$ is finite by hypothesis and that

$$F(0) = (J\Delta^{1/2}\psi, D\, J\Delta^{1/2}\psi) = (-D\Delta^{1/2}\psi, \Delta^{1/2}\psi) = -F(1).$$

Moreover $\frac{dF(\alpha)}{d\alpha} := -2\pi(\Delta^{\alpha/2}\psi, D^2\Delta^{\alpha/2}\psi)$ is negative in the interval $(0, 1)$. We can conclude that $F(0)$ is positive, hence the thesis. $\square$

Remark: The above theorem holds also on local wave-functions $\mathcal{K}$ modified by a phase, namely on $e^{i\alpha}\mathcal{K}$. The proof can be expended also in $\mathfrak{F}(\mathcal{H})$ to every $\psi \in \mathcal{D}(D)$ such that $\psi = A\Omega$, where $A \in \mathfrak{M}(I_1)$.

Proposition 3.4. *For every ψ in the domain of H, D and C, the generators of an unitary representation of $PSL(2, \mathbb{R})$ such that ψ is also contained in $\mathcal{K}_I$ where the interval $I = [a, b]$ is properly contained in I_1, with $0 < a < b < +\infty$, the subsequent inequalities hold*

$$a^2(\psi, H\psi) \leq (\psi, C\psi) \leq b^2(\psi, H\psi)\ .$$

Proof. The unitary transformation $U := \exp(iaH)$ maps the half circle I_1 in $I_a := [a, \infty)$ and $\mathcal{K}$ is mapped in the corresponding $\mathcal{K}_{I_a}$ by means of U, moreover $\mathcal{K}_I \subset \mathcal{K}_{I_a}$, then be $\psi \in \mathcal{K}_I$, $U^\dagger\psi \in \mathcal{K}$. Under this transformation

$$U^\dagger C U := C + 2aD + a^2 H, \qquad U^\dagger H U = H,$$

see [26] for details. Since by proposition 3.3 $(U^\dagger\psi, D\, U^\dagger\psi) \geq 0$ we have $a^2(\psi, H\psi) \leq (\psi, C\psi)$. The other inequality can be proved rotating I_1 by π, namely using the unitary transformation $\exp i\pi(H + C)/2$, that maps I_1 to its complement I_1', D to $-D$, C to H and vice versa. $\square$

The preceding theorem can be generalized straightforwardly along the following lines. First of all notice that the proof can be extended also in the Fock space $\mathfrak{F}(\mathcal{H})$ to every $\psi \in \mathcal{D}(D) \cap \mathcal{D}(H) \cap \mathcal{D}(C)$ such that $\psi = A\Omega$, where $A \in \mathfrak{M}(I_1)$, thus the inequalities established in Theorem 3.4 holds also in a the Fock space. Since the only important property in the preceding proof is the fact that D is positive on $\psi \in \mathcal{K}$, a second generalization arises remembering that D is also positive on wave-function multiplied by a constant phase: $e^{i\alpha}\psi$. That would be used in the next to prove the most important theory.

The last comment we give concerns the fact that there is no need of normalized wave-function in order to have theorem 3.4. Furthermore the logarithm is a monotone function and also an operator monotone function the proposition 3.4 suggests that

$$\log(a) \leq (\log\langle C\rangle_\psi - \log\langle H\rangle_\psi)/2 \leq \log(b)\ ,$$

where $\langle C\rangle_\psi = (\psi, C\psi)$, then we would like to see if it is possible to find a operator T arisen by H and C such that it commutes with D and $\log(a) \leq \langle T\rangle_\psi \leq \log(b)$ for $\psi \in \mathcal{K}_{[a,b]}$, this observation inspires the following theorem.
In the next we shall see that T is not simply $(\log C - \log H)/2$ on a suitable common domain, but it arises from another representation of the covering group of $PSL(2,\mathbb{R})$ that uses H, D and C as building blocks.

Proposition 3.5. *H^2, C^2, D^2 are local operator in $\mathcal{K}_I \cap \mathcal{D}$*

Proof. We say that some self-adjoint operator are local in a real subspace if the image of $\mathcal{K}$ under the operator is contained in $\mathcal{K}$ itself. And this is clearly the case in fact $J\Delta^{1/2}H^2\psi = H^2\psi$ □

The preceding proposition does not hold for H, D and C.

Proposition 3.6. *Let U be an unitary, non trivial, representation of $PSL(2,\mathbb{R})$ on the separable Hilbert space $\mathcal{H}$ generated by the operator $\{H, C, D\}$ which are essentially self-adjoint on a common domain $\mathcal{D}$. The following operators*

$$\widetilde{H} := \frac{H^2}{2}, \qquad \widetilde{D} := \frac{D}{2}, \qquad \widetilde{C} := \frac{H^{-1/2}CH^{-1/2}}{2}$$

are essentially self-adjoint on a domain $\widetilde{\mathcal{D}}$ of $\mathcal{H}$; moreover they satisfy the commutation relation of $sl(2,\mathbb{R})$. They generate a positive energy unitary representation $\widetilde{U}$ of the covering group of $SL(2,\mathbb{R})$ on $\mathcal{H}$.

Proof. Since the Hilbert space $\mathcal{H}$ is separable. The representation U on $\mathcal{H}$ can be decomposed in a direct sum of irreducible representations. We can consider a very component U_i of the decomposition as an irreducible representation on a $\mathcal{H}_i \subset \mathcal{H}$. Being $k \geq 1$ the highest weight of the representation U_i, every $\mathcal{H}_i$ is isomorphic to $L^2(\mathbb{R}^+, dE)$, where H, D, C take the usual form described in the appendix (8), while $\widetilde{H}, \widetilde{D}, \widetilde{C}$ take the other form (9). Consider $\widetilde{\mathcal{D}}$ the sum of analytic vectors. Then, by some Nelson result [27], $\widetilde{H}, \widetilde{D}, \widetilde{C}$ turns out to be the generators of an unitary representation of $\widetilde{SL(2,\mathbb{R})}$ the covering group $SL(2,\mathbb{R})$ on $\mathcal{H}$. □

Remark: (a) We remind that the irreducible representations of the covering group of $SL(2,\mathbb{R})$ are labelled by the lowest eigenvalue of the generator of rotation $(H + C)/2$, and the representation is positive energy if $k \geq 1/2$. Be k and $\widetilde{k}$ the lowest eigenvalues of the generators of rotations $(H + C)/2$ and $(\widetilde{H} + \widetilde{C})/2$ respectively, $\widetilde{k} = k/2 + 1/4$, since H, C, D generates an unitary representation of $PSL(2,\mathbb{R})$, $k \geq 1$ and $\widetilde{k} \geq 3/4$, hence $\widetilde{H}, \widetilde{C}, \widetilde{D}$ generate a positive energy representation of the covering group of $SL(2,\mathbb{R})$.
(b) Let us analyze the relation existing between the domain of the representations generated by the selfadjoint operator H, C and D and $\widetilde{H}$, $\widetilde{C}$ and $\widetilde{D}$ defined on suitable domains $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ on $\mathcal{H}$ as in proposition 3.6. As far as we know there is no clear connection between $\mathcal{D}$ and $\widetilde{\mathcal{D}}$, but something can be said for the domain of the quadratic forms associated with the considered operators. We remind that having a self-adjoint operator X defined on the domain

$\mathcal{D}(X)$, the associated quadratic form $(\cdot, X\cdot)$ can in principle be extended to a quadratic form $X(\cdot,\cdot)$ defined on a domain $\mathcal{Q}(X) \times \mathcal{Q}(X)$ larger then $\mathcal{D}(X) \times \mathcal{D}(X)$. Now consider $\mathcal{Q} \subset \mathcal{H}$ formed by the elements ψ of the Hilbert space such that the quadratic forms $H(\psi,\psi)$, $C(\psi,\psi)$ and $D(\psi,\psi)$, which extend $(\cdot, H\cdot)$, $(\cdot, C\cdot)$ and $(\cdot, D\cdot)$, are defined and finite. For this set, the following relation holds:

Proposition 3.7. *Be $\mathcal{Q}$ as described above, then $\widetilde{\mathcal{D}} \subset \mathcal{Q}$.*

Proof. Consider a $\psi \in \widetilde{\mathcal{D}}$ then, clearly $H(\psi,\psi) \leq (\psi, \widetilde{H}^{1/2}\psi)$ is well defined and finite. The same is true for $D(\psi,\psi) \leq (\psi, 2\widetilde{D}\psi)$ which is also finite. The only difficult part in this proof is to check that $C(\psi,\psi)$ is finite, to this end notice that

$$C(\psi,\psi) \leq (\|\widetilde{C}\psi\| \, \|\widetilde{H}^{1/2}\psi\|)^{1/2} + (\gamma\|\widetilde{D}\psi\| \, \|\widetilde{H}^{-1/2}\psi\|)^{1/2} \; ;$$

which is clearly finite hence $\psi \in \mathcal{Q}$ the thesis. □

Now for the generator of the new representation we have surprisingly a proposition similar to proposition 3.4.

Proposition 3.8. *Let $\psi \in \mathcal{K}_I \cap \widetilde{\mathcal{D}}$ where $I = [a,b] \subset I_1$ then*

$$\frac{a^2}{2}\|\psi\|^2 < (\psi, \widetilde{C}\psi) < \frac{b^2}{2}\|\psi\|^2 \, .$$

Proof. $\widetilde{C} = H^{-1/2}CH^{-1/2}/2$, then the first inequality can be proved transforming unitarily: $\varphi := U^\dagger\psi$ and $U^\dagger\widetilde{C}U$, where $U := e^{iaH}$. Notice that $\varphi \subset \mathcal{K}$ and that $U^\dagger\widetilde{C}U = H^{-1/2}(C + aD + a^2H)H^{-1/2}/2$. Since $\varphi \in \mathcal{K} \cap \widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}}$ is contained in the domain of selfadjointness of $H^{-1/2}$ and $iH^{-1/2}\varphi \in \mathcal{K} \cap \mathcal{D}$, in a similar way as in proposition 3.3

$$(\varphi, H^{-1/2}DH^{-1/2}\varphi)$$

is positive. Then $\frac{a^2}{2}\|\psi\|^2 \leq (\psi, \widetilde{C}\psi)$ holds. The other inequality can be proved rotating I_1 by π. □

Because of the previous discussion it appears clear that it is only necessary to pass to logarithm of the operator $\widetilde{C}$ in order to define an operator describing the position it being compatible with locality. To this end consider the following theorem.

Theorem 3.2. *Let T be the operator*

$$T := \frac{1}{2}\log(2\widetilde{C}).$$

It is self-adjoint on a suitable domain $\mathcal{D}(T)$ that contains $\widetilde{\mathcal{D}}$. Moreover it generates a unitary group of transformation $W(a) := e^{iaT}$ which, together with $V(t) := \Delta^{it/(2\pi)}$ generate a two dimensional Weyl Heisenberg group

Proof. $\widetilde{C}$ is positive and selfadjoint on $\mathcal{D}(\widetilde{C})$, consider its spectral measure $P(\lambda)$, then

$$T := \frac{1}{2}\int_0^\infty \log(2\lambda)\, dP(\lambda)$$

is defined on $\mathcal{D}(T) := \left\{\psi, \frac{1}{2}\int_0^\infty |\log(2\lambda)|^2 d(\psi, P(\lambda)\,\psi) < \infty\right\}$. Since by proposition 3.1 $\widetilde{\mathcal{D}} \subset \mathcal{D}(\widetilde{C}^{-1/2})$, $\widetilde{\mathcal{D}}$ is also contained in $\mathcal{D}(T)$. The proof of the second part of the theorem follows straightforwardly by noticing that there is a representation of $\widetilde{SL(2,\mathbb{R})}$ generated by $\{\widetilde{H}, \widetilde{D}, \widetilde{C}\}$. Let $U(a) := e^{ia\widetilde{C}}$ then $V(t)U(a)V(-t) = U(e^{-2t}a)$, and thus $W(a)$ with $V(t)$ generate a two dimensional Weyl Heisenberg group. □

Theorem 3.3. *Let T be defined as in theorem 3.2. Take $\psi \in \mathcal{K}_I \cap \widetilde{\mathcal{D}} \subset \mathcal{K}_{I_1}$ where I is the interval $[a, b] \subset I_1$, concerning the expectation values of T on ψ it holds*

$$\log(a)\ \|\psi\|^2 \le (\psi, T\,\psi) \le \log(b)\ \|\psi\|^2\ . \tag{7}$$

Proof. Consider

$$F(\alpha) := \frac{a^{-2\alpha}}{2\|\psi\|^2}(\psi, (2\widetilde{C})^\alpha \psi)\ .$$

Since $\widetilde{C}$ is positive and $\psi \in \widetilde{\mathcal{D}}$, by spectral properties of $\widetilde{C}$ F is a well defined smooth real function for $-1 \le \alpha \le 1$ real. Since $\psi \in \mathcal{D}(T)$, and $\log(2\widetilde{C}) = 2T$ on that domain, we have $\frac{dF}{d\alpha}(0) = -\log(a) + (\psi, T\psi)$. As a second step notice that $F(0) = 1$ while, by proposition 3.8, $F(-1) \le 1$. $\frac{d^2F}{d\alpha^2} \ge 0$ in the interval $(-1, 1)$ hence the first inequality in (7) is proved. The second can be proved in a similar way performing a rotation $U(R(\pi))$. □

The following holds by the definition of T, the action of $g \in PSL(2,\mathbb{R})$ on I and due to theorem 3.3.

Corollary 3.2. *Let $\psi \in \mathcal{K}_I$ with $\|\psi\|^2 = 1$ then $(\psi, T\psi)$ transform covariantly under unitary transformation of $PSL(2,\mathbb{R})$ that maps $I = [x, y]$ in $I_i \subset I_1$, in other words if $I_i = gI$ where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and U_g the corresponding unitary transformation:*

$$\log(x) \le (\psi, T\psi) \le \log(y) \qquad \textit{implies} \qquad \log\left(\frac{dx-b}{a-cx}\right) \le (\psi, U_g T U_g^\dagger \psi) \le \log\left(\frac{dy-b}{a-cy}\right)\ .$$

Remark: The theorem 3.3 can be generalized to every element ψ of $\mathfrak{K} := (\mathfrak{M}(I_1)\Omega \cap \{\Omega\}^\perp) \subset \mathfrak{H}$. Indeed (1) on $\mathfrak{K}$ the spectrum of $\widetilde{C}$ is continuous, furthermore Δ and J are the modular operator and conjugation in $\mathfrak{H} := \mathfrak{F}(\mathcal{H})$ for $\mathfrak{M}(I_1)$, (2) $J\Delta^{1/2}\psi = \psi$ for every $\psi \in \mathfrak{K}$. (1) and (2) are the only properties used to produce the result. This assures that T defined above is compatible with locality. Hence, since T enjoys canonical commutation relations with D, T may be interpreted as well behaved coordinate associated with the interval I_1.

4 Physical situations

In this section we introduce a concrete example where Möbius covariance arises and hence presenting a coordinate operator T for the half circle. Without losing the generality, in this cases, we assume that locality arises by construction and not intrinsically from the Möbius group (anti)-unitary representation. We shall show that real Hilbert subspace descending from intrinsic locality in the interval I contains the local wavefunction with support in I.

4.1. Coordinate operator of quantum particle on the half line. We study the quantization of a one dimensional particle associated with the so called tachyon field. This theory was proposed some years ago by Sewell [28], restricting some Minkowskian free field theory on the Killing horizon. Sometime in literature this is related with lightfront holography [29]. Recently its relation with conformal theory was put forward [30, 26]. For conformal theory on the circle see for example [14, 15, 16, 18] and reference therein. Consider the classical wave-functions ψ in $\mathcal{S}$: the set of smooth real function vanishing with every derivative at infinity. Equip $\mathcal{S}$ with the symplectic form

$$\sigma(\psi,\psi') := \int_{\mathbb{R}} \psi\partial_x\psi' - \psi'\partial_x\psi \; dx \; ,$$

which is invariant under change of coordinate x. Once the coordinate x is chosen, every wave-function can be decomposed in positive and negative frequency parts: $\psi(x) := \psi_+(x) + \overline{\psi_+(x)}$, where the positive frequency part is $\psi_+(x) := \int_0^\infty \frac{e^{-iEx}}{\sqrt{4\pi E}}\widetilde{\psi}_+(E)$. The set of ψ_+ turns out to be (dense in) a Hilbert space

$$\mathcal{H}_{\mathbb{R}} \simeq L^2(\mathbb{R}^+, dE)$$

with the scalar product $\langle\psi_+,\psi'_+\rangle := -i\sigma(\overline{\psi}_+,\psi'_+)$. As discussed in [26] on $L^2(\mathbb{R}^+, dE)$ acts a faithful (anti)-unitary representation of the Möbius group, in particular the one with lowest eigenvalues of $(H+C)/2$ equal to one, the explicit form of the corresponding generators is (8). Furthermore the action of $PSL(2,\mathbb{R})$ has a local meaning, in particular the wave-function ψ transform according to

$$U_g\psi(x) := \psi\left(\frac{ax+b}{cx+d}\right) - \psi\left(\frac{a}{c}\right) \qquad g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in PSL(2,\mathbb{R}),$$

under $g \in PSL(2,\mathbb{R})$ while J is a reflection $J\psi(x) = \psi(-x)$.

There are two ways to build real local subspaces of wave-functions, one that arises from geometrical considerations and the other from the group properties of $PSL(2,\mathbb{R})$. In the former case, the wave-function in $\mathcal{S}_I$ are defined as the elements of $\mathcal{S}$ with support contained in I. In the latter case we use the set $\mathcal{K}_I$ defined in (5). Let $\varphi \in \mathcal{S}_I$ then $J_I\Delta_I^{1/2}\varphi_+ = \varphi_+$ holds. In some sense, with a little abuse of notation, we can say that for every I, $\mathcal{S}_I \subset \mathcal{K}_I$. The result found in the previous section holds true also for the elements of $\mathcal{S}_I$: Consider $\varphi \in \mathcal{S}_I$ where the interval $I = [a,b] \subset \mathbb{R}^+$, we get $(\varphi_+, D\varphi_+) \geq 0$ by proposition 3.3, $a^2(\varphi_+, H\varphi_+) \leq (\varphi_+, C\varphi_+) \leq b^2(\varphi_+, H\varphi_+)$ by proposition 3.4. It is furthermore possible to define the operator T related with the half line coordinate defined as in 3.2

$$T := \frac{1}{2}\log(2\widetilde{C}) \ ; \qquad \text{where} \qquad \widetilde{C} := -\frac{d^2}{dE^2}$$

is a selfadjoint operator on $\mathcal{H}_{\mathbb{R}}$. Finally by theorem 3.3, for every $\phi := P\hat{W}(\psi)\Omega$, where P is a projector on $\mathfrak{K}$ holds $\log a \leq (\phi, T\phi) \leq \log b$ if $\|\phi\| = 1$.

Since $\widetilde{C}$ can be extended on $\mathfrak{F}(\mathcal{H}_{\mathbb{R}})$ and it has has zero eigenvalues only on $\alpha\Omega$, the operator $T := \log(2\widetilde{C})/2$ turns out to be self-adjoint on a domain $\mathcal{D}$ dense in the sub Hilbert space $\{\Omega\}^{\perp}$, finally $\log(a)\|\phi\|^2 \leq (\phi, T\phi) \leq \log(b)\|\phi\|^2$ is valid for $\phi \in \mathfrak{A}(I)\Omega \cap \{\Omega\}^{\perp}$.

We conclude, surprisingly, that T, which is defined *intrinsically* by means of the representation of $PSL(2,\mathbb{R})$, is self-adjoint on a suitable domain and defines a modular coordinate which is compatible with locality. It is true even if local properties of T are not manifest considering its action on a wave-function $\varphi \in \mathcal{S}$.

5 Final comments

We have seen that, in case of Möbius covariant theories, an observable representing the coordinate of a particle within an interval I arises in a natural way by the covariance structure itself. Furthermore this observable turns out to be compatible with the localization properties defined intrinsically as in [1, 2, 3]. In the last section we have also presented a concrete example where the found operator is compatible with the standard localization scheme.

It is worth to stress that the proposed coordinate can be defined in every Möbius invariant theory, even if the group of symmetry is hidden, namely has not a geometric action on the element of the Hilbert space. In those cases the physical meaning of the proposed coordinate operator and also of the intrinsic locality needs to be studied carefully. This happens for example in free field theories on two dimensional Minkowski spacetime, where a hidden $PSL(2,\mathbb{R})$ symmetry is present, and the usual locality properties differ from the intrinsic locality described above. Furthermore the intrinsic locality turns out to be compatible with the locality shown by a dual theory defined on a null surface [28, 26]. Similar situation arises considering dual theories on null surfaces as black hole horizon or null infinity of asymptotically flat spacetime, where the locality, that arises on the null surface, differs from the one present in the spacetime.

There are situation in which the modular dilations $\Delta^{it/(2\pi)}$ represent time translations, as for example in particular thermal theories. In those cases the found coordinate operator which is self-adjoint, represent a time. We stress that, since the corresponding energy D is not bounded from below, there is no need of generalizing the concept of observables by means of POVM as in the standard situation [22, 23, 24]. The procedure is very similar to the definition of a time operator in thermal theories [25], where, because of the unboundedness of the energy, a selfadjoint time operator can be defined.

Finally we would like to emphasize once again the intrinsic character of the proposed coordinate operator, because its definition involves only the generators of the group of symmetry. This suggests that space and/or time could arise as derived concept, namely as target space of particular observables representing spacetime coordinate.

Acknowledgments. The author is grateful to R. Brunetti, S. Doplicher, K. Fredenhagen and V. Moretti for stimulating discussions and interesting suggestions. This work has been funded by Provincia Autonoma di Trento by the project FQLA, Rif. 2003-S116-00047 Reg. delib. n. 3479 & allegato B, and partially supported by a grant from GNFM-INdAM (Istituto Nazionale Di Alta Matematica) under the project *"Olografia e spazitempo asintoticamente piatti: un approccio rigoroso"*.

Appendix

A $PSL(2,\mathbb{R})$ representations

The positive energy irreducible representations π_k of the covering group of $SL(2,\mathbb{R})$ are labelled by a real number $k \geq 1/2$ which coincides also with the lowest eigenvalue of the generator of rotation $(H+C)/2$ [31, 32]. Every irreducible representation π_k on the Hilbert space $L^2(\mathbb{R}^+, dE)$ is generated by the following operators

$$H := E, \qquad D = -i\sqrt{E}\frac{d}{dE}\sqrt{E}, \qquad C = -\sqrt{E}\frac{d^2}{dE^2}\sqrt{E} + \frac{k^2-k}{E}, \tag{8}$$

which are defined on a suitable domain $\mathcal{D}^k$ formed by linear combination of the set of vectors

$$Z_m^{(k)}(E) := \sqrt{\frac{\Gamma(m-k+1)}{E\,\Gamma(m+k)}}(2\beta E)^k e^{-\beta E} L_{m-k}^{(2k-1)}(2\beta E);$$

where $L_n^{(k)}(x)$ are the generalized Laguerre polynomials, and β is a positive constant. H, C, D satisfy the $sl(2,\mathbb{R})$ commutation relations:

$$[H, D] = iH, \qquad [C, D] = -iC, \qquad [H, C] = 2iD.$$

The action of H, C, D is closed on D^k. Moreover $Z_m^{(k)}$ are analytic vectors for $H^2+C^2+D^2$ then by some Nelson results [27] they are essentially self-adjoint namely their self-adjoint extensions $\overline{H}$,$\overline{D}$,$\overline{C}$ are unique and exists a representation of the covering group of $SL(2,\mathbb{R})$ on $\mathfrak{H}$ generated by $\overline{H}$,$\overline{D}$,$\overline{C}$. In the paper we have discarded the over line and we have indicated by H, D, C the self-adjoint operators.

Remark: (a) In particular, if k is integer π_k turns out to be a representation of $PSL(2,\mathbb{R})$ which is faithful if $k = 1$. If instead k is half-integer π_k is a representation of $SL(2,\mathbb{R})$ which is faithful for $k = 1/2$.
(b) Since the vacuum is invariant under π_k it can be extended to a representation on the Fock space. Consider the complex conjugation J, its action on H, D, C is as follows:

$$JHJ = H, \qquad JDJ = -D, \qquad JCJ = C.$$

The found representation of $PSL(2,\mathbb{R})$ with the above defined J forms an irreducible representation of the Möbius group.

A.1. Another $sl(2,\mathbb{R})$ representation. Consider, on the one particle Hilbert space $L^2(\mathbb{R}^+, dE)$ the following operators

$$\widetilde{H} := \frac{E^2}{2}, \qquad \widetilde{D} = -i\frac{1}{2}\sqrt{E}\frac{d}{dE}\sqrt{E}, \qquad \widetilde{C} = -\frac{d^2}{dE^2} + \frac{k^2 - k}{E^2}. \tag{9}$$

The set $\widetilde{\mathfrak{D}}^k$ formed by linear combination of the set of vectors

$$\widetilde{Z}_m^{(k)}(E) := \sqrt{\frac{2\,\Gamma(m-k+1)}{E\,\Gamma(m+k)}}(2\beta E^2)^k e^{-\beta E^2} L_{m-k}^{(2k-1)}(2\beta E^2);$$

turns out to be a dense set of common analytic vectors, then they are essentially self-adjoint. Furthermore, since they satisfy the $sl(2,\mathbb{R})$ commutation relations, they generate a positive energy representation of the covering group of $SL(2,\mathbb{R})$ labelled by $k \geq 1/2$. Here k has the same meaning as the k of the preceding sections (remark (a) above).

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